Brane Configurations of BPS Domain Walls for the $\mathcal{N} = 1^*$ $SU(N)$ Gauge Theory

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Abstract

We study supersymmetric domain walls in $\mathcal{N} = 1$ $SU(N)$ gauge theory with 3 massive adjoint representation chiral multiplets. This theory, known as $\mathcal{N} = 1^*$, can be obtained as a massive deformation of $\mathcal{N} = 4$ Yang-Mills theory. Following Polchinski and Strassler, we consider the string dual of this theory in terms of spherical 5-branes and construct BPS domain walls interpolating between the many vacua. We compare our results to field theoretic domain walls and also find that this work is related to the physics of expanded "dielectric" branes near zero radius.

1 Introduction

Recently, Polchinski and Strassler [?] found the string theory dual to an $\mathcal{N} = 1$ gauge theory with adjoint matter, which can be obtained by giving masses to three chiral superfields in the $\mathcal{N} = 4$ $SU(N)$ gauge theory. They called this field theory $\mathcal{N} = 1^*$ and obtained the string dual by generalizing the AdS/CFT correspondence [?]. On the string side of the duality, the N D3-branes that source the AdS geometry arrange themselves into 5-branes with N units of D3 charge due to the RR 6-form corresponding to the mass perturbation in the CFT (as first discussed in [?]). Following Myers [?], they suggested that the D3-branes form a 5-brane extended in the dimensions of the D3-branes with the other two dimensions wrapped on an S^2. The numbers and types of 5-branes in the configuration correspond to specific vacua in the gauge theory, and their vacuum (coordinate) radii and orientations are determined by a superpotential on the brane. In particular, the totally Higgsed vacuum corresponds to a single D5-brane, various Coulomb vacua correspond to multiple D5-branes, and the confining and oblique confining vacua correspond respectively to a single NS5- or $(1, k)$5-brane (for $1 \leq k < N$). For specifics of the brane description, see [?]; other studies of the $\mathcal{N} = 1^*$ theory include [?, ?, ?, ?, ?, ?].

Because the $\mathcal{N} = 1^*$ $SU(N)$ theory has a large (but finite for finite N) number of vacua, there should generically be domain walls interpolating between pairs of those vacua. In supersymmetric theories with domain walls, there has been much interest in finding BPS domain wall solutions – domain walls that preserve some supersymmetry. In particular, for gauge theories, studies of BPS domain walls include [?, ?, ?, ?, ?, ?, ?, ?, ?]; see [?] for further references on BPS domain walls. Recently, [?, ?] studied BPS domain walls in the $\mathcal{N} = 1^*$ theory from the field theory perspective.

In this paper, we investigate the brane configurations corresponding to BPS domain walls that interpolate between different vacua in the gauge theory. On the string side, the 5-branes bend from one vacuum state to the other, and when branes with non-zero net 5-brane charge intersect, another 5-brane fills the S^2 at the intersection [?]. Polchinski and Strassler [?] discussed two examples of domain walls in the small coupling limit and compared the vacuum state superpotentials and domain wall tensions to exact field theoretic calculations of [?, ?], finding agreement within their approximations.

We expand those results to finite string coupling and construct domain wall brane configurations. In section 2, we review the 5-brane actions of [?] and establish some approximations. In section 3, we find conditions necessary for the mechanical equilibrium and supersymmetry of the 5-brane junctions, and section 4 uses the general results of sections 2 and 3 to confirm that the BPS bound for the domain wall tension on the string side matches the field theoretic bound. In section 5, we construct a number of BPS domain walls and note interesting examples. Finally, in section 6, we summarize our results and discuss their relation to the body of research on BPS domain walls in field theory. In both sections 5 and 6, we will discuss the supersymmetric minimum in the brane potential at vanishing 5-brane sphere size [?, ?] and its relation to the brane picture of BPS domain walls. In the end, though, we will have to confess ignorance as to the meaning or existence of a zero size state for the 5-brane spheres.

While preparing this paper, we became aware of work by C. Bachas, J. Hoppe, and B. Pioline [?] that has some overlap with this work. Specifically, they find BPS domain wall configurations interpolating between Coulombic vacua (and the Higgs vacuum) within field theory. We discuss these domain walls in section 5.1 and compare our results to those of [?] in section 6.3.

2 Brane Actions

In this section, we will discuss the action that describes the 5-brane bending for the domain walls. Through the rest of this paper, we will follow the conventions of Polchinski and Strassler [?], working to leading order in their small parameter, the ratio of 5-brane to D3 charge. In doing so, we ignore the near-shell corrections to the metric and supergravity fields, so we take the dilaton to be constant and the Einstein frame metric to be equal to the string frame metric.

First, we rederive the action for brane bending given in equation (126) of [?]. The S^2 part of the 5-brane twists and contracts (or expands) as we pass from one vacuum state to another (moving in, without loss of generality, the x^1 direction with translational invariance in the x^2 and x^3 directions). To start, we note that the induced metric in the directions parallel to the D3-branes is

$$G_{\mu\nu}(\text{induced}) = G_{\mu\nu} + G_{mn}\partial_\mu x^m \partial_\nu x^n \ . \tag{1}$$

The Dirac-Born-Infeld action of a (c,d) 5-brane is therefore

$$\begin{aligned} S &= -\frac{\mu_5}{g|M|^2}\int d^6\xi \left[-\det\left(|M|Z^{-1/2}\eta_{\mu\nu} + |M|Z^{1/2}\delta_{mn}\partial_\mu x^m\partial_\nu x^n\right)\right. \\ &\qquad \left. \times \det\left(|M|G_\perp + 2\pi\alpha'\mathcal{F}\right)\right]^{1/2} \ . \end{aligned} \tag{2}$$

Here, $M = c\tau + d$ with coupling $\tau = \theta/2\pi + i/g$ for the (c,d) brane, and $G_\perp$ is the metric on the wrapped S^2 of the 5-brane [**?**].

Under the assumption of slow bending (small derivatives of brane position), the factor in the x^{0-3} directions gives

$$|M|^2\left(Z^{-1} + \frac{1}{2}\eta^{\mu\nu}\partial_\mu x^m\partial_\nu x^m\right) = |M|^2\left(Z^{-1} + (2\pi\alpha')^2\eta^{\mu\nu}\partial_\mu\bar\phi\partial_\nu\phi\right) \ . \tag{3}$$

In the last step, we have substituted the scalar field ϕ introduced in [**?**] to describe the size and orientation of the S^2. For a real unit vector e^i on the S^2, ϕ is defined by

$$(2\pi\alpha')\phi e^{1,2,3} = \frac{1}{\sqrt{2}}\left(x^{4,5,6} + ix^{7,8,9}\right) \ . \tag{4}$$

This may seem like a somewhat suspect expansion, given that Z diverges at the brane. However, it corresponds to the self-reaction of a charge, which should be ignored, as discussed in [**?**]. We can think of the 5-brane as built up from infinitesimal 5-branes, each of which acts as a probe brane to the rest of the geometry. Since each probe action is independent of Z, the expansion follows for the full 5-brane.

The rest of the action follows as in [**?**]; we can expand both the S^2 determinant and the Chern-Simons action in powers of the D3 charge, which is assumed to dominate. Then the action for n D3 charges (integrated over the S^2) is

$$S = -\int d^4x\left[\frac{n\mu_3}{g}(Z^{-1} - Z^{-1}) + \frac{n}{2\pi g}\eta^{\mu\nu}\partial_\mu\bar\phi\partial_\nu\phi + \frac{2\pi g}{n}|W_\phi|^2\right] \tag{5}$$

where $W_\phi = 1/(2\pi g)(mn\phi + i2\sqrt{2}M\phi^2)$ is the derivative of the superpotential

$$W = \frac{1}{2\pi g}\left(\frac{mn}{2}\Phi^2 + i\frac{2\sqrt{2}}{3}M\Phi^3\right) \ . \tag{6}$$

A few comments are in order. First, the leading Z^{-1} terms from the Dirac-Born-Infeld and Chern-Simons actions cancel, as they should for parallel D3-branes. However, when we consider the tension of the 5-brane, it is precisely the D3 tension $n\mu_3/g$ that dominates. Also, the vacuum configuration of the 5-branes occur at the roots of the potential $|W_\phi|^2$, where the configuration and superpotential are

$$\phi_v = \frac{imn}{2\sqrt{2}M}\,,\; W_v = -\frac{m^3n^3}{96\pi g M^2}\,. \tag{7}$$

For a configuration with multiple branes, the superpotential is summed over the branes, and the vacuum configuration of each brane is unaffected by the other branes [**?**].

Since we hope to find supersymmetric domain walls (with configurations varying only in the x^1 direction), it is useful to write the action (eqn 5) as

$$S = -\int dx \left(\frac{2\pi g}{n} \left| \frac{n}{2\pi g}\partial_1\phi - \Omega\overline{W_\phi}\right|^2 + \Omega\partial_1\overline{W} + \overline{\Omega}\partial_1 W \right)\,, \tag{8}$$

with Ω a complex phase. From the above, we can see that brane bending that follows the “BPS equation”

$$\frac{n}{2\pi g}\partial_1\phi = \Omega\overline{W_\phi} \tag{9}$$

both satisfies the equations of motion and preserves supersymmetry. In this case, the domain wall tension due to the brane bending is given by the surface terms. Further, each 5-brane in the domain wall must bend with the same phase Ω for supersymmetry to be preserved. Also, since the BPS equations for different 5-branes (in the same vacuum state, say) are decoupled except for having the same value of Ω, each 5-brane bends independently. We emphasize that each 5-brane follows its own BPS bending equation independently of any other 5-branes present and note that we will consider only domain walls with BPS brane bending.

At some point in the transition from one vacuum to another with a differently charged 5-brane(s), charge conservation requires that an additional 5-brane be present, as discussed in [**?**]. If the domain wall interpolates between, for example, a (c,d)5-brane and a (c',d')5-brane, then the third 5-brane has charge $(c'-c, d'-d)$. Further (taking the vacua to change along the x^1 direction), this extra 5-brane extends in the $x^{2,3}$ directions and fills the S^2 at the intersection of the two “vacuum branes.”

The DBI part of the action of this 5-brane ball is (at lowest order in the perturbations)

$$S = -\frac{\mu_5}{g|M|^2}\int d^6\xi\sqrt{-\det\left(|M|G_{ab}\right)} = -\frac{\mu_5|M|}{g}\int d^3x\,\frac{4\pi r_0^3}{3}\,, \tag{10}$$

where r_0 is the radius of the S^2 and $M = c\tau + d$ is the 5-brane charge as above. The metric factors Z have canceled because the 5-brane is extended in 3 dimensions each of factor $Z^{-1/2}$ and $Z^{1/2}$. Then the domain wall tension due to this ball-filling brane is

$$\tau(\text{ball}) = \frac{\mu_5|M|}{g}\frac{4\pi r_0^3}{3} = \frac{2\sqrt{2}|\phi_0|^3|M|}{3\pi g} \tag{11}$$

for ϕ_0 the configuration for all the vacuum branes at the brane junction.

3 Force-Balancing Equations

We consider in this section the force-balancing equations at the 5-brane junction, which are needed to maintain mechanical equilibrium and preserve supersymmetry. These are analogous to conditions derived in [**?**, **?**] but are quantitatively different due to the D3 tension here. We specialize to a static domain wall along x^1 (hereafter x) and translationally invariant in $x^{2,3}$. We consider any number of 5-branes labeled by the index i with 5-brane charges M_i and n_i units of D3 charge each approaching the junction from smaller x and 5-branes labeled by j approaching from larger x.

For a 5-brane junction to occur, the 5-branes must first intersect. On first glance, this means that the brane configuration variables ϕ must be the same for all the vacuum 5-branes at the junction. However, we note that ϕ and $-\phi$ describe the same sphere, so there could be a brane junction between, for example, two 5-branes with opposite ϕ. We deal with this case by noting that the action (8) (and BPS equation (9)) is invariant under $\phi \to -\phi$, $M \to -M$, so a 5-brane is equivalent to the corresponding anti-5-brane with opposite configuration variable. Physically, taking $\phi \to -\phi$ leaves the S^2 invariant while reversing its orientation and therefore the 5-brane charge. Then we can always choose 5-brane charges at a junction so that all the 5-branes have the same configuration variable, which takes the value ϕ_0 at the junction (though we might need to consider different junctions separately in each domain wall).

The force-balancing conditions should be derived in an inertial reference frame. In this flat metric, the D3- and 5-brane tensions are simply the flat space values and give the force per proper area on the junction in the directions of extent of those branes. Thus, for n D3-branes on an S^2 of coordinate radius r, the total D3 tension on a unit area of the S^2 is

$$\frac{\mu_3}{g}\frac{n}{4\pi r^2 Z^{1/2}} \equiv \frac{\mu_3}{g}\rho\,. \tag{12}$$

Technically, we are working slightly away from the 5-branes (so Z is finite) – this substitutes for building up the domain wall out of infinitesimal charges – in an orthonormal basis aligned along the coordinate axes. We consider the north pole of the spherical brane junction, which is in the x^6, x^9 plane with $\arg(x^6 + ix^9) = \arg(\phi_0) \equiv \alpha$. Henceforth, we denote $x^{6+i9} \equiv x^6 + ix^9$.

Now consider a unit vector $v_{i,j}$ along each vacuum brane in the direction out of the junction. In the small bending approximation,

$$v_{i,j}^{\mathbf{x}} = \mp 1\ ,\ v_{i,j}^{\mathbf{6+i9}} = \mp Z^{1/2}\partial_x x_{i,j}^{6+i9} \tag{13}$$

where bold indices are the orthonormal frame indices and the signs are for the indices i, j respectively. A ball-filling 5-brane has a tangent vector

$$v^{\mathbf{6+i9}} = -e^{i\alpha} \tag{14}$$

leaving the north pole of the brane junction. Then mechanical equilibrium requires

$$\frac{\mu_5}{g}\left|\Delta M\right| v + \frac{\mu_3}{g}\left(\sum_i \rho_i v_i + \sum_j \rho_j v_j\right) = 0 \tag{15}$$

for $\Delta M = \sum_j M_j - \sum_i M_i$. Equation (15) gives the conditions

$$\sum_i n_i = \sum_j n_j \tag{16}$$

in the **x** direction and

$$|\Delta M|\, e^{i\alpha} = \frac{1}{4\pi\sqrt{2}\,|\phi_0|^2}\left(\sum_j n_j \partial_x \phi_j - \sum_i n_i \partial_x \phi_i\right) \tag{17}$$

in the $\mathbf{6} + \mathbf{i9}$ directions.

The first of these equations simply gives conservation of D3 charge across the junction. For BPS brane bending according to equation (9), D3 charge conservation requires the linear terms from the potential to cancel. Then we find simply that

$$e^{i3\alpha} = -i\Omega\overline{\Delta M}/\,|\Delta M|\ . \tag{18}$$

This implies that the discontinuity of the superpotential at the junction is

$$\Delta W_0 = \frac{i}{2\pi g}\frac{2\sqrt{2}}{3}\Delta M \phi_0^3 = \Omega|\Delta W_0|\,. \tag{19}$$

Note, however, if there is no 5-brane ball at the junction, there is no condition on the phase of ϕ_0.

To this point, we have considered only positive D3-brane charge, as negative D3 charges would break supersymmetry and add a high energy cost. We can also see this from mechanical equilibrium considerations; if we allow negative D3 charges, equation (16) becomes

$$\sum_i |n_i| = \sum_j |n_j| \tag{20}$$

which conflicts with charge conservation at the junction. Thus, we conclude that negative D3 charges will not appear in any BPS domain wall.

4 Domain Walls: Generalities

In this section, we discuss the BPS bound for the domain wall tension and confirm that it matches the BPS bound from field theory for a domain wall interpolating between any two vacua of the $\mathcal{N} = 1^*$ theory. We also make some observations that will allow us to discuss some specific explicit domain wall solutions in the following section.

As discussed above, we consider domain walls which follow the BPS equation (9) for bending of the 5-branes. Combining the BPS equation and its conjugate, we find

$$\overline{\Omega}\partial_x \phi\, W_\phi = \Omega \partial_x \overline{\phi}\, \overline{W_\phi}\,, \tag{21}$$

or that the imaginary part of $\overline{\Omega}W$ is conserved. This implies that, up to discontinuities in W at brane junctions, the BPS trajectory follows a straight line in the complex W plane with a tangent vector of complex phase $\pm\Omega$ (in the direction of increasing x). Due to the force-balancing condition equation (19), the discontinuous jump in W at a brane junction also has the phase Ω, so the trajectory of the superpotential, including discontinuities, is then a straight line directly from $W(x=-\infty)$ to $W(x=\infty)$ (ie, between the two vacuum values). Thus we have $\Omega = \pm\Delta W/|\Delta W|$. In the case that there is no ball-filling 5-brane at a junction, there is no constraint from mechanical equilibrium, but there is also no discontinuity in the superpotential, so we still have a straight line. We should note that, in field theory, the superpotential trajectory would also follow this straight line but without discontinuities.

Now consider the domain wall tension. From the action (8) and BPS equation (9), the tension from brane bending is

$$\int dx \left(\overline{\Omega}\partial_x W + \Omega\partial_x \overline{W}\right) = 2\mathrm{Re}\left(\overline{\Omega}\Delta W\right) \tag{22}$$

along any single 5 brane. We see that $\Omega = -\Delta W/|\Delta W|$ gives a negative tension, which comes from a *positive definite* Hamiltonian, so that choice is unphysical. Taking $\Omega = \Delta W/|\Delta W|$, the brane bending contributes $2|\Delta W(\mathrm{bending})|$ to the domain wall tension. Further, twice the discontinuity in the superpotential at a 5-brane junction is equal in magnitude to the tension of the 5-brane ball at that junction:

$$2|\Delta W_0| = \frac{4\sqrt{2}}{3(2\pi g)}|\Delta M|\,|\phi_0|^3 = \tau(\mathrm{ball}) \tag{23}$$

(compare to equation (11)). Thus, assuming $\Delta W/|\Delta W| = +\Omega$, we find the BPS bound for domain wall tension

$$\tau_{DW} = 2\,|\Delta W| \tag{24}$$

in agreement with field theoretic results since [**?**] found that the vacuum superpotentials (eqn (7)) match the exact field theoretic results (within our approximations). Since the BPS bound on the domain wall tension follows from the supersymmetry algebra (see, for example, [**?**] for a review of the central charges of domain walls), it is not surprising to find the same bound; however, it is satisfying to see new physics give the same tension.

5 Domain Walls: Specifics

In this section, we will discuss explicit domain wall solutions and comment on them. We will begin by discussing domain walls between vacua with only one type of 5-brane (for example, D5-branes only on both the left and the right), which we can discuss analytically because the 5-brane spheres keep the same orientation throughout the domain wall. We will proceed to consider domain walls between vacua with single 5-branes of different types and will finally discuss some examples of domain walls between vacua with multiple 5-branes of different types. Throughout, we take the mass m to be positive.

5.1 Single Type of 5-Brane Charge

Here, we consider only domain walls between vacua with only one type of 5-brane charge given by $M = c\tau + d$. As discussed above, each 5-brane has a vacuum configuration and superpotential given by equation (7) (inserting the appropriate value of n for each 5-brane). Therefore, we see that all the 5-branes have the same orientation and same phase of the superpotential. For definiteness, we will always take domain walls with the magnitude of the superpotential increasing from negative to positive x, which gives

$$\Omega = -\exp\left(-2i\arg M\right) . \tag{25}$$

Additionally, we can rotate the phase of the brane configuration variables to $\phi = i\exp(-i\arg M)\psi$ with vacuum $\psi_v = mn/2\sqrt{2}|M|$. Then the BPS equation becomes

$$\frac{1}{m}\partial_x\psi = \overline{\psi} - \frac{\overline{\psi}^2}{\psi_v} . \tag{26}$$

Without losing generality, we can take all the vacuum 5-branes to be D5-branes from this point forward; the only difference would be the size of ψ_v.

To begin, we consider domain walls in which ψ remains real for all the 5-branes, so the BPS equation has the following solutions:

$$\begin{aligned}
\psi_<(x,x_0,n) &= \psi_v\frac{e^{m(x-x_0)}}{1+e^{m(x-x_0)}} \text{ for } 0<\psi<\psi_v\,,\ \text{all } x && (27)\\
\psi_>(x,x_0,n) &= \psi_v\frac{e^{m(x-x_0)}}{e^{m(x-x_0)}-1} \text{ for } \psi>\psi_v\,,\ x>x_0 && (28)\\
\psi_-(x,x_0,n) &= \psi_v\frac{e^{m(x-x_0)}}{e^{m(x-x_0)}-1} \text{ for } \psi<0\,,\ x<x_0\,. && (29)
\end{aligned}$$

In the above, x_0 is an integration constant, and the vacuum value ψ_v is the appropriate value for a 5-brane with n D3 charges. The two positive solutions are plotted in figure 1. It is important to note that none of these solutions goes to ψ_v as $x \to -\infty$, so all the vacuum D5-branes at negative x must remain in their vacuum configurations until they reach a brane junction (for the real solutions that we consider here). We will come back to this point later. Also, the force-balancing condition (eqn. (18)) for $\psi > 0$ implies that any 5-brane junction should have more D5-branes on the left (lesser x) than on the right (or an equal number) in a BPS configuration.

Another useful solution to the BPS equation is that for a sphere of D3-branes with no 5-brane charge (henceforth a "zero-5-brane"), which bends according to the BPS equation with $M = 0$. With Ω and ψ defined as above, the BPS equation for a zero-5-brane and its real solution are

$$\begin{aligned}
\partial_x\psi &= m\overline{\psi} && (30)\\
\psi_{(0)}(x,\psi(0)) &= \psi(0)e^{mx}\,. && (31)
\end{aligned}$$

A few words are necessary about the zero-5-brane configuration. In a vacuum state, the D3 branes with no 5-brane charge would collapse to a point. This situation seems physically similar to the minimum of the

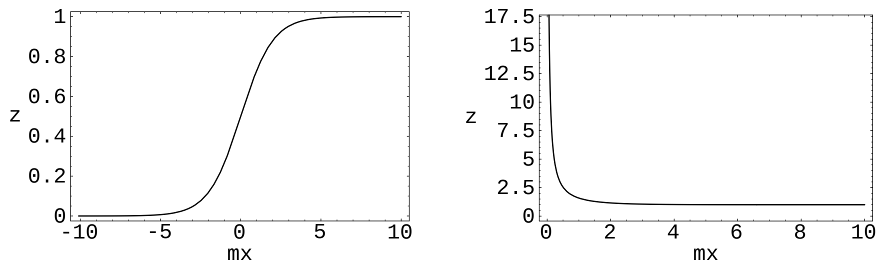


Figure 1: These are the two positive, real solutions to equation (26) given by equation (27) on the left and equation (28) on the right. The vertical axis in both cases is $z = \psi/\psi_v$.

potential $|W_\phi|^2$ at $\phi = 0$ for any 5-brane charge, which is usually considered an unphysical minimum [**?**, **?**]. However, inside a domain wall, we need not be concerned whether the zero-5-brane corresponds to a physical vacuum or not. We will discuss this point further below.

Now we can construct explicit domain wall solutions. Because we are taking all the vacuum 5-branes to be D5s, we are discussing domain walls between various Coulomb vacua and the Higgsed vacuum – Coulomb vacua are vacua with multiple D5-branes and $U(1)^k$ or $SU(k)$ gauge symmetry unbroken, while the Higgs vacuum is a single D5-brane with all gauge symmetry broken. In the following domain walls, we will define $\psi_V = mN/2\sqrt{2}|M|$, the vacuum configuration for a single 5-brane vacuum, for convenience.

The first domain wall we will consider is between a vacuum with two D5 branes with fN and $(1-f)N$ D3 charges each $(1/2 \leq f < 1)$ and a single D5 with all N D3 charges. In this domain wall, the smaller D5 on the left remains in its vacuum state for $x < 0$, where there is a junction with a ball-filling D5-brane and a zero-5-brane that follows the solution $\psi_{(0)}(x, (1-f)\psi_V)$. The larger vacuum brane on the left stays in its vacuum state for $x < (1/m)\ln(f/1-f)$, where it enters a junction with the other end of the zero-5-brane and the vacuum brane from the right. The single D5-brane on the right follows $\psi_<(x, 0, N)$ for $x > (1/m)\ln(f/1-f)$. This configuration is shown in figure 2. We should note that when $f = 1/2$, there is no zero-5-brane.

Another domain wall of interest is that between two Coulomb vacua, such as the vacuum discussed above, with D3 charges given by f_1 on the left and f_2 on the right. Because the magnitude of the superpotential of such a vacuum is proportional to $f^3 + (1-f)^3 = 1 - 3f + 3f^2$, we see that we should take $f_2 > f_1$ to have the larger superpotential at positive x. The BPS domain wall corresponding to these vacua is shown in figure 3(a). It has no 5-brane balls and has brane junctions between the two smaller D5s and between the two larger D5s, which are connected by a zero-5-brane. If we choose to have the junction of the two smaller branes at $x = 0$, the small D5 on the right bends according to $\psi_>(x, x_0, (1-f_2)\psi_V)$ for $x > 0$, the zero-5-brane bends according to $\psi_{(0)}(x, (1-f_1)\psi_V)$ for $0 < x < y =$

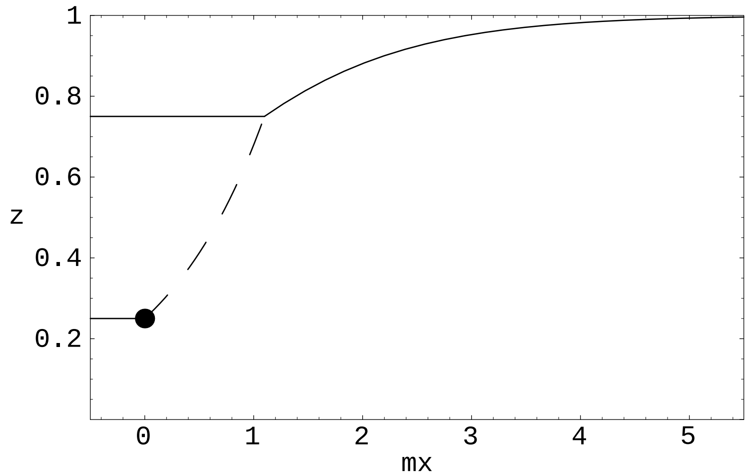
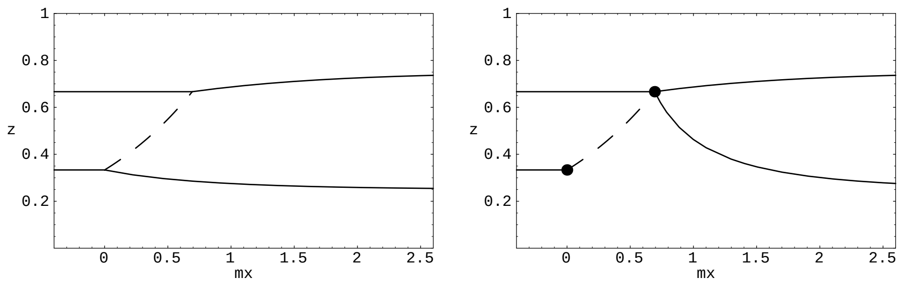


Figure 2: The domain wall between two D5s and one D5. In this case, $f = 3/4$, and in all figures, $m = 1$ and $N = 12000$. The vertical axis is $z = \psi/\psi_V$. The dashed line is a zero-5-brane, and the dot is a ball-filling D5-brane. (These conventions are standard in figures 3 and 4.)

$(1/m)\ln(f_1/(1-f_1))$, and the large D5 on the right bends according to $\psi_<(x, x_0, f_2\psi_V)$ for $x > y$. Here, $x_0 = -(1/m)\ln((1-f_1)/(f_2-f_1))$.

Some interesting physics arises if we consider other domain walls interpolating between these two vacua. If we just consider BPS brane bending (as in eqns. (27,28)), it appears that we could have BPS domain walls such as those shown in figure 3(b), where the small D5 on the right does not connect to the small D5 on the left. In fact, it appears that there is a continuous family of such domain walls with a D5-brane branching off of a zero-5-brane. However, these domain walls are *not* BPS and are not mechanically stable (despite the fact that they have BPS brane bending) because the brane junction involving the small vacuum brane on the right *does not satisfy the force-balancing condition* (18). These domain walls actually have a higher tension than the BPS bound due to “backtracking” of the domain wall trajectory in the complex W plane. Essentially, the smaller D5-brane at positive x has to bend too much. So, given one of these non-BPS domain walls, the D5-brane balls, which are anti-branes of each other, are free to attract (because there is a continuous family of domain walls) and annihilate, leaving the BPS solution, classically at least.

A final illustrative case to consider is a domain wall interpolating between Coulomb vacua with two and three D5-branes respectively. If none of the three 5-branes is larger than both of the two 5-branes, then a BPS domain wall can be constructed similar to the first domain wall discussed above (see figure 4 for some BPS domain walls). However, in cases where one of the three D5-branes is larger than both of the D5- branes in the other vacuum, there appears to be no BPS domain wall constructed out



(a) BPS Domain Wall

(b) Unstable, Non-BPS Domain Wall

Figure 3: Two domain wall configurations between two Coulomb vacua. In this case, $f_1 = 2/3$ and $f_2 = 3/4$. There is also a continuous family of non-BPS domain walls interpolating between these two cases, which allow the non-BPS domain wall to decay into the BPS domain wall.

of real solutions to the BPS equation. If, for example, the three D5-brane vacuum has a smaller magnitude superpotential, the domain wall would require a zero-5-brane with negative D3 charge, which we have see are ruled out. Similarly, if the three D5 vacuum has a greater magnitude superpotential, there would be at least one 5-brane junction with more D5-branes at larger x, violating the condition of equation (18).

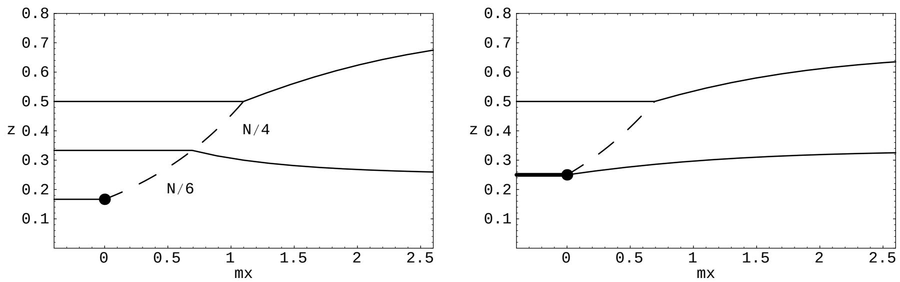
Might there be BPS domain walls between these vacua with complex values of ψ? Considering the BPS equation (26) for $\psi = \psi_1 + i\psi_2$ (for $\psi_{1,2}$ real), we have

$$\frac{1}{m}\partial_x\psi_1 = \psi_1 - \frac{1}{\psi_v}\left(\psi_1^2 - \psi_2^2\right) \qquad (32)$$

$$\frac{1}{m}\partial_x\psi_2 = \left(2\frac{\psi_1}{\psi_v} - 1\right)\psi_2 \, . \qquad (33)$$

The boundary conditions on the vacuum branes at positive x require that they do not twist (that ψ remains real). To see this, consider a perturbation of equations (32,33) for a single 5-brane around ψ_v at $x = +\infty$ and note that such a perturbation vanishes at ∞ only if it is real. Then equation (33) requires the solution to remain real. For the vacuum branes at negative x, any imaginary perturbation is unstable (because $\psi_1 \to \psi_v$ as $x \to -\infty$), leading one to suspect that there is no BPS domain wall between these vacua. Figure 5 shows a vectorfield of $(1/m)\partial_x(\psi_1\ \psi_2)$ in the $\psi_1,\ \psi_2$ plane, which seems to indicate that an imaginary perturbation would not flow back to the real axis. It is possible that some other type of 5-brane plays a role in these "complex" domain walls; however, such a configuration, if it exists, would be difficult to find.

It is, however, not altogether surprising that some pairs of vacua do not have BPS domain walls; consider one vacuum with D5-branes of D3 charge $(12, 16, 19, 24)N$ D3 and another vacuum with D5-branes of D3



(a) The three D5-branes carry 1/6, 1/3, and 1/2 of the D3 charges; the two carry 1/4 and 3/4. Note that the two zero-5-branes carry different amounts of D3 charge.

(b) The three D5s carry 1/4, 1/4, and 1/2 of the D3 charges; the two carry 1/3 and 2/3.

Figure 4: BPS domain walls for three D5-branes going to two D5 branes. In both cases, the vacuum with three 5-branes has a smaller magnitude superpotential than the vacuum with two. In part 4(b), the two smaller 5-branes at negative x are chosen to carry the same D3 charge. Thick lines indicate two D5-branes.

charges respectively $(9, 10, 15, 27)N$. These two vacua have the same superpotential, so any BPS domain wall between them would be tensionless. On the other hand, the vacua are not identical, so there must be some positive tension due to brane bending in any domain wall. Thus, there is no BPS domain wall[1]. It is worth noting, though, that higher order effects in the 5-brane charge or in the $1/N$ expansion might or might not lift the degeneracy of these vacua and permit a BPS domain wall. Also, it is known that $\mathcal{N} = 1$ supersymmetric $SU(N)$ theories with matter in the fundamental representation do not have BPS domain walls if the matter mass is too large [**?**, **?**, **?**][2]. In any event, we can conjecture that a condition for a pair of Coulomb vacua to be connected by a BPS domain wall is that they have unequal superpotentials and that the none of the branes in the vacuum with more branes be larger than the largest brane in the other vacuum.

5.2 Domain Walls Between Single 5-Branes

In this section, we consider domain walls between vacua with one 5-brane. As discussed in section 4, we know that the BPS trajectory along the domain wall follows a straight line in the complex superpotential plane, even including discontinuities at 5-brane junctions, and $\Omega = \Delta W/|\Delta W|$. This allows us to solve the full complex BPS equation (9) numerically;

[1] Thanks to I. Bena and M. Patel for discussions of this point.

[2] We need not worry that the value of the mass will alter the spectrum of BPS domain walls in our case; since we work with a deformation of a conformal theory, m is the only length scale.

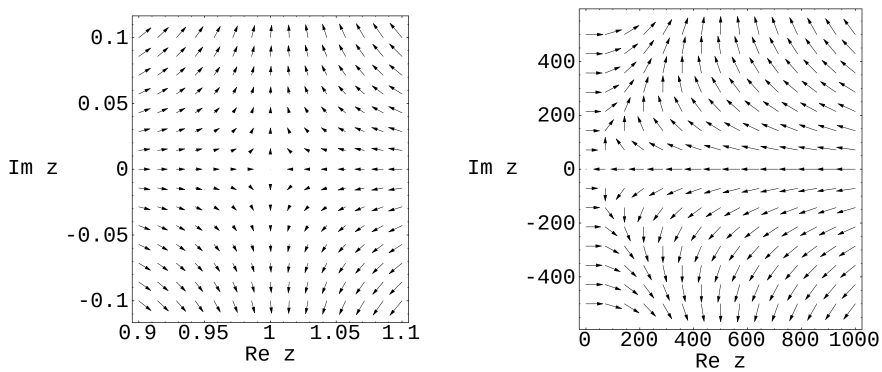


Figure 5: A plot of $\partial_{mx}\psi_{1,2}$ at two axis scales; axes are given by $z = (\psi_1 + i\psi_2)/\psi_v$. It does not seem that any solution leaving the horizontal axis from $\psi_1/\psi_v = 1$ would ever return to the real axis.

we can solve the cubic superpotential W for the configuration variable ϕ and plot the trajectory in configuration space as a function of W on the line segment from $W(x \to -\infty)$ to $W(x \to \infty)$. If we do this for the two vacuum branes, we can find the configuration ϕ_0 where they intersect to find the BPS domain wall. For single 5-brane vacua, D3 charge is automatically conserved at the brane junction, and, for the specified value of Ω, the phase condition (19) is satisfied up to a sign, which we check numerically.

One issue is that there are three solutions for ϕ as a function of W because the superpotential is cubic. However, only two of these approach the vacuum configuration ϕ_v as the superpotential goes to the vacuum value W_v (see eqn. (7)). Those two solutions correspond to the two solutions $\psi_>$ and $\psi_<$ discussed above, essentially. We can choose whichever solution will intersect with the other 5-brane.

(An additional check on numerical solutions is provided by the argument of $\partial_x\phi$ at the vacuum state. By considering linear perturbations of the BPS equation (9) around the vacuum, we find

$$\partial_x\delta\phi_1 = -m(\Omega_1\delta\phi_1 + \Omega_2\delta\phi_2) \tag{34}$$

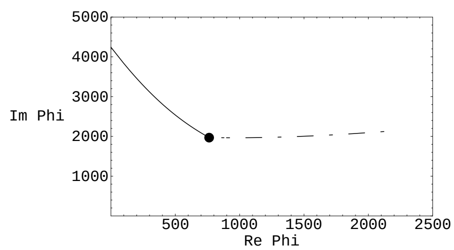
$$\partial_x\delta\phi_2 = -m(\Omega_2\delta\phi_1 - \Omega_1\delta\phi_2) \tag{35}$$

for $\delta\phi_{1,2}$ and $\Omega_{1,2}$ real and imaginary parts respectively. Only one of the eigenvectors has $\delta\phi \to 0$ for $x \to \infty$ (or $-\infty$, as desired), so this gives the phase of $\partial_x\phi$. This checks numerically.)

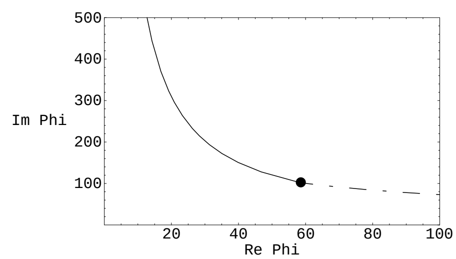
The first case to consider are for a domain wall interpolating between a D5-brane and a $(1,1)$5-brane ($(1,k)$5-brane vacua are sometimes called oblique confining vacua, while the NS5-brane vacuum is the confining vacuum). Figure 6 shows graphs of the brane configuration parameter ϕ for a domain wall interpolating between a D5-brane and a $(1,1)$5-brane with several values of the string coupling g. For small g, the $(1,1)$5-brane looks like an NS5-brane, while for large g, it looks like a D5-brane. Thus, for small g, most of the domain wall tension is from brane bending, while, for large g, it is mostly from the 3-ball 5-brane. The NS5-$(1,1)$ domain wall is S-dual to this case, and the other D5/NS5- to $(1,k)$5-brane domain walls are similar. All these configurations satisfy the force-balancing condition, and they are the only BPS domain walls found between each pair of vacua.

The domain wall interpolating between a D5-brane vacuum and an NS5-brane vacuum is an interesting case. For a value of the RR scalar $\theta > 0$, we can find the domain wall just as before (see figure 7(a)). For $\theta < 0$, on the other hand, the D5 and NS5 branes intersect with opposite values of ϕ, meaning that we can interpret the domain wall as interpolating between an anti-D5-brane and an NS5-brane (or between a D5- and an anti-NS5-brane) (see figure 7(b)). For $\theta = 0$, though, the domain wall does not seem to exist. We can understand this from the BPS equation solutions found in section 5.1; for the appropriate value of $\Omega = 1$ in this case, both of the 5-branes follow the solution $\psi_<$ (equation (27)). (See equations (34,35) to see that the appropriate perturbations are along the real and imaginary axes for the NS5 and D5 branes respectively.) Thus, the NS5(D5)-brane stays on the real (imaginary) ϕ axis, and they intersect only at the origin, which their solutions reach only in an infinite distance. Figure 8 shows a vector field of the BPS equation flow for both the D5- and NS5-brane, and they seem not to intersect. Another argument that the D5- and NS5-brane (with $\theta = 0$) have no BPS domain wall connecting them is that such a domain wall would seem to violate the 5-brane/anti-5-brane symmetry of the physics. This second argument should hold even when higher order corrections are taken into account. It is also notable that the tension of this domain wall would scale as N^4 in the 't Hooft limit, indicating that it may not exist [**?**].

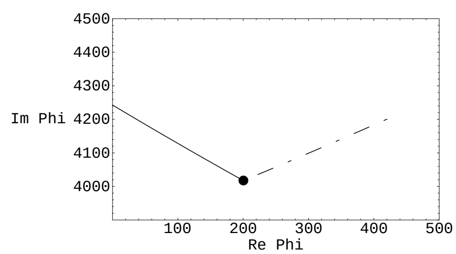
Physically, we seem to have a case of BPS spectrum restructuring in different regions of parameter space, as was discussed recently by [**?**]. In their language, $\theta = 0$ is the curve of marginal stability for the domain wall between the D5- and NS5-branes. For positive θ, the "stable" BPS object is a domain wall between D5- and NS5-branes, while for negative θ the BPS domain wall is between an anti-D5- and an NS5-brane. At $\theta = 0$, if the analogy with the results of [**?**] holds, the domain wall supposedly becomes a composite of widely separated BPS solitons, presumably the domain walls between the D5/NS5-branes and the "vacuum" at $\phi = 0$. As mentioned above, most sources consider the $\phi = 0$ vacuum to be unphysical, and we certainly do not have a good description of its physics if it does exist, since corrections due to the 5-brane charge and quantum string physics would play a large role there. Another possibility is that near shell corrections or quantum effects glue together the D5- and NS5-branes at a small size, leaving a very thick but finite size domain wall between the Higgs and confining vacua. That would be a specific example of the transformation of 5-brane charge at zero size conjectured in [**?**].



(a) String coupling $g = 1$.

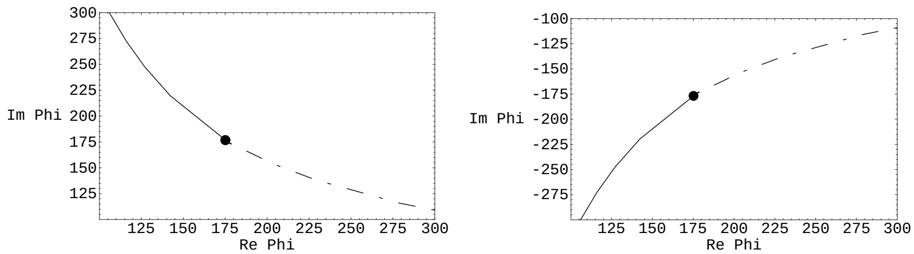


(b) String coupling $g = 0.1$.



(c) String coupling $g = 10$.

Figure 6: The BPS domain walls interpolating between a D5-brane and a $(1,1)$5-brane. The axes are the real and imaginary ϕ axes (as in figure 7); the D5 is a solid line, and the $(1,1)$5-brane is the dot-dashed line. The dot represents a ball-filling NS5-brane. Part 6(b) is cropped to show the junction more clearly. The RR scalar θ is set to zero.



(a) $\theta = (0.01)2\pi$, D5 to NS5

(b) $\theta = (-0.01)2\pi$, anti-D5 to NS5

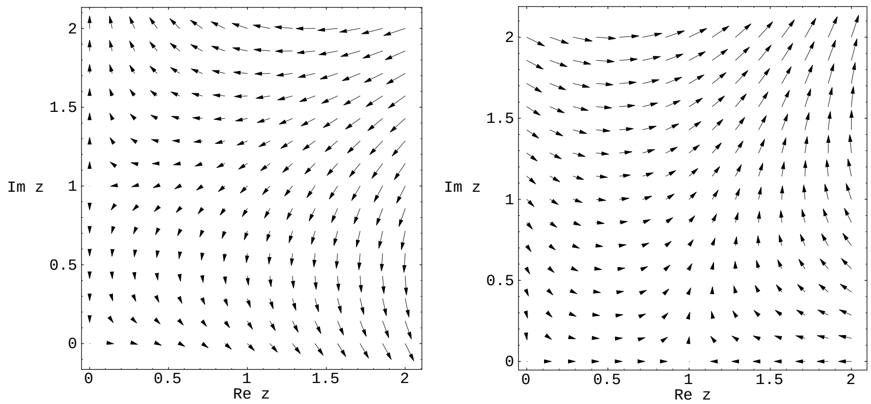
Figure 7: BPS domain walls between the Higgs and confining vacua with $\theta \neq 0$. The D5-brane is a solid line, the NS5 dot-dashed, and $g = 1$. Large dots are ball-filling 5-branes. Note that these plots are the complex conjugates of each other.

5.3 Multiple 5-branes of Different Types

To find domain walls between the most general pairs of vacua, namely those with several 5-branes of different types, we will typically need to use the numerical methods of the section 5.2 above. These are applicable basically without modification, since the superpotential is additive over different 5-branes and the BPS brane-bending equations are decoupled for the separate branes. It is typically difficult to find explicit solutions in the most general cases, because three or more curves in the complex plane will generically not all intersect at a point. Such general BPS domain walls, if they exist, require extra 5-branes, similar to the zero-5-branes in section 5.1 but without exact solutions to the BPS equation that make it possible to find the domain walls. Here, we will discuss two cases with extra symmetry that allow us to find BPS domain walls.

First, we can discuss domain walls connecting a D5-branes to $(1,k)$5-branes. Assuming that all of the 5-branes in each of the vacua have the same number of D3 charges (that is, not all of the gauge symmetry is broken), these domain walls are more or less rescaled versions of those discussed above. These domain walls are uncomplicated because the vacua are essentially single 5-branes with multiple 5-brane charges.

The case of a D5 and NS5 each with $N/2$ D3-brane charges going to a $(1,1)$5-brane also has a special symmetry; it is self-S-dual for $\theta = 0$ and $g = 1$, and there is no ball-filling 5-brane. This domain wall is shown in figure 9 for $g = 1$. For other values of the string coupling, the three vacuum branes never all have the same ϕ, so any domain wall would be more complicated, as in more general cases. However, one would expect that BPS domain walls would exist at least for a range of g near 1 because one does exist at that special value of the coupling.



(a) D5 brane flow

(b) NS5 brane flow

Figure 8: Vector flow fields for the D5/NS5-brane bending with $\Omega = 1$, $g = 1$, $\theta = 0$. The axes are given by $z = \phi/|\phi_v|$, where the vacuum state spheres have the same size. Note that the D5 flow could conceivably approach the real axis, but, once it converted to an NS5-brane, it would be swept away from the real axis.

6 Summary

6.1 Domain Walls

In summary, we have discussed BPS domain walls in the string theory dual of the $\mathcal{N} = 1^*$ $SU(N)$ theory. Using a small-bending approximation for the vacuum state 5-branes, we found, as in [**?**], that the 5-branes bend independently of each other (that is, independently of the configuration variables of the other 5-branes, including the metric factor Z). We were also able to establish, using conditions for mechanical equilibrium, that the BPS bound for the domain wall tension is the same in the string theory as in the field theory dual, given that the vacuum superpotentials are the same.

We then discussed domain wall solutions for a number of pairs of vacua. For domain walls between Coulomb (and Higgs) vacua, we gave analytic solutions for BPS brane bending and demonstrated a general construction of BPS domain walls. We were also able to find conditions in which BPS domain walls do not seem to exist, and we gave an example of a non-BPS domain wall and a mechanism through which it can decay classically to a BPS domain wall. We also found numerical BPS domain wall solutions interpolating between the Higgs and oblique confining vacua, as well as for some special cases with multiple 5-branes.

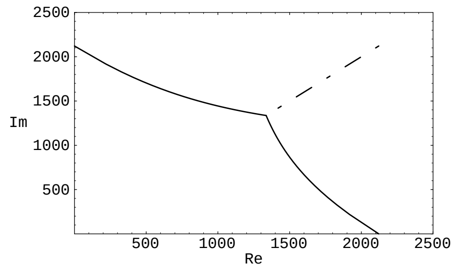


Figure 9: The domain wall for a D5 and NS5 going to a $(1,1)$5-brane with $g = 1$ and $\theta = 0$. There is no 5-brane ball.

6.2 Zero Radius 5-Brane Spheres

Importantly, we also encountered physics involving the zero radius configuration of the 5-brane spheres. In section 5.1, we used zero-5-branes to carry D3 charge without carrying 5-brane charge; the vacuum state for such a zero-5-brane would have a collapsed sphere, if it exists. We reemphasize that a vacuum state for the zero-5-brane need not be physical for zero-5-branes to occur in non-vacuum configurations, such as domain walls. We also found that the domain wall between the Higgs (D5) and confining (NS5) vacua with zero RR scalar seems not to exist in this approximation. One interpretation of this is that the zero-size state (which is a minimum of the potential) does exist and that $\theta = 0$ is a curve of marginal stability on which the D5/NS5 domain wall decomposes into two domain walls. However, recent literature is of the opinion that the zero-size state is unphysical (as supported by the string exclusion principle [**?**][3]). In that case, it is possible that effects due to the 5-brane charge or quantum string physics connect the D5- and NS5-branes at a finite size. At this time, the physics behind such a transition is not understood and remains a point of interest for future study.

It may also be interesting to compare the possible zero-size vacuum of brane physics to the controversial chirally symmetric Kovner-Shifman vacuum of supersymmetric $SU(N)$ theory [**?**]. For example, both appear as minima of effective potentials that describe the theory around a particular vacuum state (for a discussion of the Veneziano-Yankielowicz Lagrangian for $\mathcal{N} = 1$ Yang-Mills theory [**?**] and an extension of it, see [**?**]) which may or may not be valid near the zero-size or chirally symmetric vacuum respectively. Assuming they exist, the zero-size and Kovner-Shifman vacua do have some at least superficial similarities; first, they both have a zero superpotential, as opposed to the (oblique) confining vacua. They also would both have BPS domain walls connecting them to the oblique confining vacua in which one real field varies (in the Kovner-Shifman case, that is the gluino condensate [**?**]). It would be interesting to calculate the gluino condensate in the zero-size vacuum through the AdS/CFT correspondence in order to see if it is also chirally symmetric, but – if that

[3]Thanks to J. McGreevy for a discussion of this point.

vacuum exists – it lies outside of the approximations of [?]. However, both of these vacua are generally considered to be unphysical (see [?] for a recent critique of the Kovner-Shifman vacuum). Perhaps a physical understanding of why one of these vacua fails to exist (or a determination that it does indeed exist) would shed light on the physics of the other.

6.3 Comparison to Field Theoretic Results

It is also possible to connect our results to previously known field theoretic results. First, we will compare our results to the recent work of [?], which found BPS domain wall configurations interpolating between Coulomb and Higgs vacua in the $\mathcal{N}=1^*$ theory, as in section 5.1 here. In general, our results agree (including the solutions to the BPS equation, when given), but there are two comments to be made here. One is that not every pair of Coulomb vacua with different values of the superpotential have a BPS domain wall, as [?] speculated. It may be that the formal condition (4.18) in [?] is equivalent to our condition and that cases without BPS domain walls only arise for large enough N. The other comment is that the configuration (2.14) of [?] does not connect the Coulomb vacua to the confining vacuum, as we have discussed extensively above. Because [?] considered only the classical vacuum structure, they neglected quantum effects that give the confining vacuum a gluino condensate and non-zero superpotential.

Recent studies of BPS domain walls in supersymmetric gluodynamics in the large N limit have stressed that the BPS tension between adjacent oblique confining vacua (such as an NS5- and $(1,1)$5-brane) scales as N, while the natural energy density scales as N^2, leading to the conclusion that the domain wall thickness must vanish as $1/N$ [?, ?, ?, ?]. These studies have also noted that such a scaling is precisely that expected for a D-brane, in line with a suggestion by Witten [?] that a domain wall would act as a D-brane for the QCD string. As [?] stated, the domain walls considered here demonstrate that domain walls in the $\mathcal{N}=1^*$ theory are indeed D-branes. If we consider, for example, the BPS domain wall between an NS5-brane and a $(1,1)$5-brane in the 't Hooft limit, then the vacuum states differ only at order $1/N$, so the dominant contribution to the tension is from the ball-filling D5-brane, which has a vanishing thickness and on which precisely the correct flux tube can end [?].

6.4 Future Directions

Finally, we should note a few future directions to take. In terms of string physics, an understanding of the domain wall between D5- and NS5-branes or a definitive determination whether it exists would be important in understanding the physics of D-brane spheres at small radius in RR-form backgrounds. With the motivation of studying domain walls, this work could be extended to BPS domain wall junctions in the $\mathcal{N}=1^*$ theory [?], which have been of increasing interest in the literature [?, ?]. Another direction might be to eliminate the small-bending approximation, which would correspond to finding a more general form of the Kähler potential for the configuration variable ϕ but would make finding explicit solutions much more difficult.

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